Anisotropy in homogeneous rotating turbulence

José Gaite
Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, 28006 Madrid, Spain
(Received 20 June 2003; published 25 November 2003)

The effective stress tensor of a homogeneous turbulent rotating fluid is anisotropic. This leads us to consider the most general axisymmetric four-rank "viscosity tensor" for a Newtonian fluid and the new terms in the turbulent effective force on large scales that arise from it, in addition to the microscopic viscous force. Some of these terms involve couplings to vorticity and others are angular momentum nonconserving (in the rotating frame). Furthermore, we explore the constraints on the response function and the two-point velocity correlation due to axisymmetry. Finally, we compare our viscosity tensor with other four-rank tensors defined in current approaches to nonrotating anisotropic turbulence.

DOI: 10.1103/PhysRevE.68.056310 PACS number(s): 47.27.−i, 47.32.−y

I. INTRODUCTION

The properties and applications of rotating fluids constitute an important area of fluid mechanics [1]. In particular, the anisotropy consequent to the rotation has been a very important subject. For example, we have the classical Proudman-Taylor theorem, which says that, in the limit of fast rotation, the flow is so strongly anisotropic that it actually becomes two dimensional. Turbulence in the presence of uniform rotation, which is called rotating turbulence, is an example of anisotropic turbulence and an area of active research [1–4].

From a theoretical point of view regarding symmetry, the classical theory of fully developed turbulence assumes the maximal possible symmetry, namely, symmetry under translations and rotations, so it applies to ideal *homogeneous and isotropic* turbulence. However, in various situations such a high symmetry is not realistic and one must consider less symmetric turbulent states. The next most symmetric state is still homogeneous but the isotropy reduces to axisymmetry, that is, the full rotation group reduces to the group of rotations around a particular axis. The archetype of homogeneous turbulence with axisymmetry is rotating turbulence (naturally, the symmetry axis is the rotation axis).

It was shown in Ref. [5] that perturbation theory for the randomly forced rotating Navier-Stokes equation generates anisotropic effective forces, in particular, the *nondissipative* force $\boldsymbol{\Omega}\times\nabla^2\boldsymbol{u}$. This suggests that one should find the complete set of allowed force terms. In this regard, it is useful to define the effective viscosity, which is a tensorial function of $\boldsymbol{\Omega}$ (reproducing the known perturbative results for isotropic turbulence as $\Omega\to 0$). Beyond perturbation theory (or any method of solution), our intention here is to study from first principles the consequences of axisymmetry in rotating turbulence.

The possibility of anisotropy in the velocity correlation functions has been considered before in nonrotating fluids [6–12]. So, in these references, the anisotropy was attributed to other causes: existence of a mean flow or anisotropic forcing. In fact, in a *homogeneous* fluid the existence of mean flow effects, that is, the dependence of properties of the flow on its mean velocity, would contradict Galilean invariance. The flow can only depend on global kinematical features that involve accelerations, such as in a uniformly rotating fluid. A homogeneous but anisotropic forcing will induce anisotropy in the velocity field (the axial case is studied in Ref. [9]); but the physical origin of this anisotropic forcing and, therefore, the extent of the scale range affected by it are not clear. We think that rotating turbulence is a more natural example of anisotropic turbulence and with more physical applications. Moreover, this type of anisotropic turbulence has distinctive features (as was pointed out in Ref. [5]) because $\boldsymbol{\Omega}$ is an axial vector. Indeed, the force $\boldsymbol{\Omega}\times\nabla^2\boldsymbol{u}$ or other terms of the same type would not be allowed if isotropy were broken by a polar vector as in Ref. [9].

We remark that the characterization of the effective (or eddy) viscosity as a four-rank tensor has already appeared in the literature. For example, in Ref. [7] the authors show that a multiscale method applied to the Navier-Stokes equation linearized with respect to a weak large-scale flow precisely produces an effective viscosity tensor if the basic fluctuating flow is not isotropic. However, as commented above, to determine the form of this tensor, one needs an explicit mechanism that breaks isotropy and preserves homogeneity. Otherwise, the basic assumptions and, in particular, axisymmetry, are not justified.

On the other hand, since $\boldsymbol{\Omega}$ is an axial vector, the effective viscosity tensor in rotating turbulence has distinctive features: for example, it has a pair-antisymmetric piece (which generates the above-mentioned force) [5]. In addition, it will be shown here that a general treatment of the effective viscosity tensor in rotating turbulence requires new terms that couple to the vorticity or that are angular momentum nonconserving (in the rotating frame) and, therefore, are forbidden in anisotropic nonrotating turbulence.

We shall first review the fluid equations in the rotating frame and the conditions for turbulence; we emphasize the transition from small-scale isotropic turbulence to large-scale anisotropic turbulence. Next, we introduce the viscosity in the standard manner [13] but *without* recourse to isotropy, which is replaced by only axisymmetry. All the components of the resulting four-rank tensor are determined with group theory arguments [14]. From this tensor we obtain the additional anisotropic force terms. Once seen the axisymmetry constraints on the viscosity tensor, we impose axisymmetry on the response function or (two-point) velocity correlations. In particular, the large-scale response function is related with the viscosity tensor. Finally, we try to connect the viscosity tensor with other four-rank tensors introduced in some current approaches to nonrotating anisotropic turbulence.

II. EQUATIONS OF MOTION IN A ROTATING FRAME AND TURBULENCE

The hydrodynamical equations for a fluid with density field $\rho(\boldsymbol{x},t)$, velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, pressure $P(\boldsymbol{x},t)$ in a frame rotating with constant angular velocity $\boldsymbol{\Omega}$ are

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho \boldsymbol{u})=0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}=-\frac{1}{\rho}\boldsymbol{\nabla}P-2\boldsymbol{\Omega}\times u-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x})+\boldsymbol{f}, \tag{2}$$

where $\boldsymbol{f}$ accounts for an additional acceleration due to friction (which vanishes if $\partial_i \boldsymbol{u}=0$) and a homogeneous and isotropic external forcing, usually random (or periodic as in Ref. [7]), which serves for keeping the total kinetic energy constant.

We assume that the fluid is incompressible, with constant density, so the continuity equation becomes $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$. So if we define $p=P/\rho$ every reference to the density disappears, and we have two equations for the two unknowns $\boldsymbol{u}$ and p. To solve for p, it is useful to separate Eq. (2) into independent longitudinal and transverse equations. Since $\boldsymbol{u}=\boldsymbol{u}_L+\boldsymbol{u}_T$ (such that $\boldsymbol{\nabla}\times\boldsymbol{u}_L=\boldsymbol{\nabla}\cdot\boldsymbol{u}_T=0$) and $\boldsymbol{u}_L$ identically vanishes, the longitudinal equation becomes just a constraint relating p with spatial derivatives of $\boldsymbol{u}$, namely,

$$p=\frac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{x})^2-\frac{1}{\nabla^2}[\partial_i(u_j\partial_j u_i)+2\epsilon_{ijk}\Omega_j\partial_i u_k]. \tag{3}$$

Solving for p, the equation for $\boldsymbol{u}=\boldsymbol{u}_T$ is

$$\frac{\partial \boldsymbol{u}}{\partial t}+\mathcal{P}[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}]=-\mathcal{P}(2\boldsymbol{\Omega}\times\boldsymbol{u})+\boldsymbol{f}, \tag{4}$$

where the projection operator $\mathcal{P}$ onto transverse (or solenoidal) fields is given by

$$\mathcal{P}=\mathbf{1}-\boldsymbol{\nabla}\frac{1}{\nabla^2}\boldsymbol{\nabla}. \tag{5}$$

In Eq. (4), if $\boldsymbol{u}$ is transverse so is $\boldsymbol{f}$ and vice versa. We call Eq. (4) the transverse rotating fluid equation. If we substitute for $\boldsymbol{f}$ an isotropic viscous force, it becomes the transverse rotating Navier-Stokes equation. Note that the transverse rotating fluid equation (4) is *translation invariant* (assuming that $\boldsymbol{f}$ is homogeneous), in contrast with Eq. (2). Therefore, its solutions are homogeneous velocity fields and, furthermore, one can make use of the Fourier transform.

Homogeneous rotating turbulence

The homogeneous rotating turbulent state is defined by a velocity field with large fluctuations but such that the mean velocity is negligible in the rotating frame. Let us see how to characterize this state in terms of nondimensional parameters and how it is related with the homogeneous and isotropic turbulent state.

Since we have the rotation velocity as additional parameter, we can define two nondimensional parameters, namely, the Reynolds and Rossby numbers. While the Reynolds number $\mathrm{Re}=UL/\nu$ measures the relative importance of the nonlinear and viscosity terms in the Navier-Stokes equation, the Rossby number $\mathrm{Ro}=U/(L\Omega)$ measures the relative importance of the nonlinear and Coriolis terms in the rotating Navier-Stokes equation (U is a reference velocity or the variation of the velocity over the length L that characterizes the system size). In principle, $\mathrm{Ro}\gg 1$ indicates that rotation effects are negligible and, vice versa, $\mathrm{Ro}\ll 1$ indicates that they are dominant. However, the latter condition, namely, dominance of rotation effects over nonlinear and viscous (and even dynamic) effects leads to the situation in which only the linear Coriolis force is relevant, giving rise to extreme two dimensionalization of the flow (as in the Proudman-Taylor theorem) but without turbulence. It is necessary that the *two* numbers Ro and Re play a role in specifying the regime of interest, that is to say, the regime with rotation effects ($\mathrm{Ro}\gg 1$) *and* turbulence ($\mathrm{Re}\gg 1$). Or one may introduce the Ekman number $\mathrm{Ek}=\nu/(\Omega L^2)$ (in addition to Ro), which is the ratio of the Rossby number to the Reynolds number and measures the relative importance of the viscosity and Coriolis terms [1]. Then one must demand $\mathrm{Ek}\ll\mathrm{Ro}$ in addition to $\mathrm{Ro}\ll 1$.

To clarify the preceding condition, let us consider relevant length scales. First, let us recall the role of the dissipation scale. In ordinary homogeneous and isotropic turbulence, K41 theory [8] makes the dissipation rate per unit mass ε the basic quantity and introduces the dissipation scale $\lambda=(\nu^3/\varepsilon)^{1/4}$. Using Ω instead of ν, we can form with ε the length scale $\ell=(\varepsilon/\Omega^3)^{1/2}$. If we begin with small Ω (for fixed ε) such that $\ell\gg L$, rotation effects must be negligible all over the fluid system of characteristic length L. Therefore, the precise condition for neglecting Ω is $\Omega\ll(\varepsilon/L^2)^{1/3}$ (equivalent to $\ell\gg L$). Given that $(L/\lambda)^{4/3}=\mathrm{Re}\gg 1$, the parameter $(\varepsilon/L^2)^{1/3}/\Omega=\nu/(\Omega L^2)(L/\lambda)^{4/3}=\mathrm{Ek}\,\mathrm{Re}=\mathrm{Ro}$, so the condition for neglecting Ω is just $\mathrm{Ro}\gg 1$. As Ω grows and, therefore, ℓ diminishes such that $\ell<L$, rotation effects become appreciable. We then have one scale range with rotating turbulence, namely, between ℓ and L, and another with isotropic turbulence, namely, between λ and ℓ. The latter range holds as long as $\lambda<\ell$, that is, $\Omega<\sqrt{\varepsilon/\nu}$. As ℓ becomes smaller than λ, the rotation effects dominate over the nonlinear effects and the flow becomes strongly two dimensional.

The interesting values of Ω are such that there are the two scale ranges, respectively, with isotropic turbulence on small scales and anisotropic turbulence on larger scales. Of course, this happens when $\lambda\ll\ell\ll L$. Then the viscosity or correlation functions on scales between λ and ℓ are essentially isotropic whereas the effective viscosity or correlation functions on scales between ℓ and L are axisymmetric.

III. THE AXISYMMETRIC EFFECTIVE VISCOSITY TENSOR

To introduce the viscosity tensor, it is convenient to follow the general reasoning [13] which starts by writing the fluid equation in local conservative form as

$$\rho \frac{\partial u_i}{\partial t} = \frac{\partial \Pi_{ij}}{\partial x_j}, \quad \Pi_{ij} = -\rho u_i u_j + T_{ij}, \tag{6}$$

and finds the deviatoric part of the stress tensor

$$T_{ij} = -P\ \delta_{ij} + \sigma_{ij} \tag{7}$$

due to internal relative motion (viscosity) from general principles. The first principle is that the velocity gradient is small, which allows one to consider only *first derivatives* of the velocity. Next, the viscous stress tensor σ_{ij} is taken proportional to the velocity gradient and, furthermore, its antisymmetric components (vorticity) are excluded, so that the stress is proportional to the rate of strain $u_{mn} = \partial_{(m} u_{n)} = (\partial u_n/\partial x_m + \partial u_m/\partial x_n)/2$ (this characterizes *Newtonian fluids*) [15]. The following crucial assumption is *isotropy*, which leads to the existence of only two proportionality constants (shear and bulk viscosities). As we cannot make this assumption here, we are left with just the proportionality relation

$$\sigma_{ij} = \eta_{ijmn} u_{mn}, \tag{8}$$

such as in the analogous relation in the theory of elasticity that expresses that the stress is proportional to the strain [16]. Therefore, the symmetry properties of the tensor η_{ijmn}, which we call the "viscosity tensor," are similar to the ones of the elastic modulus tensor, namely, symmetry under exchange of indices within the first and second pairs of indices and, in addition, symmetry under exchange of the first and second pairs of indices (pair symmetry). However, we shall further allow for *pair antisymmetry*; namely, we write η_{ijmn} as a sum of a pair-symmetric (S) and a pair-antisymmetric (A) part [5]:

$$\eta_{ijmn} = \tfrac{1}{2}(\eta_{ijmn} + \eta_{mnij}) + \tfrac{1}{2}(\eta_{ijmn} - \eta_{mnij}) \equiv \eta^S_{ijmn} + \eta^A_{ijmn}. \tag{9}$$

So, generically, the viscosity tensor has 36 independent components, of which 21 belong to the pair-symmetric part η^S_{ijmn} and 15 belong to the pair-antisymmetric part η^A_{ijmn}.

The axial symmetry of the equations of motion reduces the number of independent components of both η^S_{ijmn} and η^A_{ijmn}. The 21 components of the generic pair-symmetric tensor can be divided into two sets with 15 and 6 components, respectively, the former corresponding to the totally symmetric tensor. The respective components are constructed in the Appendix as linear representations and called ${}_{15}S$ and ${}_{6}S$. Further imposing axisymmetry, the pair-symmetric tensor can be constructed from Ω_i and δ_{ij} as

$$\begin{aligned}\eta^S_{ijmn} = {} & a_1(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + a_2(\Omega_i\Omega_j\delta_{mn} \\ & + \Omega_m\Omega_n\delta_{ij} + \Omega_i\Omega_m\delta_{jn} + \Omega_j\Omega_m\delta_{in} + \Omega_i\Omega_n\delta_{jm} \\ & + \Omega_j\Omega_n\delta_{im}) + a_3\Omega_i\Omega_j\Omega_m\Omega_n + a_4\delta_{ij}\delta_{mn} \\ & + a_5(\Omega_i\Omega_j\delta_{mn} + \Omega_m\Omega_n\delta_{ij}).\end{aligned} \tag{10}$$

There are five independent components, to which we attach scalars $a_1, \ldots, a_5$ (which can depend on Ω^2). In comparison with the form given in Ref. [5], this expression has been arranged so that the three first tensors (with coefficients a_1, a_2, a_3) are totally symmetric in their indices.

The generic pair-antisymmetric tensor has 15 components, constructed in the Appendix as the linear representation ${}_{15}S'$. The pair-antisymmetric tensor with axisymmetry needs, in addition to Ω_i and δ_{ij}, the totally antisymmetric tensor ϵ_{ijk} and is

$$\begin{aligned}\eta^A_{ijmn} = {} & b_1\Omega_q(\epsilon_{qim}\delta_{jn} + \epsilon_{qin}\delta_{jm} + \epsilon_{qjm}\delta_{in} + \epsilon_{qjn}\delta_{im}) \\ & + b_2\Omega_q(\epsilon_{qim}\Omega_j\Omega_n + \epsilon_{qin}\Omega_j\Omega_m + \epsilon_{qjm}\Omega_i\Omega_n \\ & + \epsilon_{qjn}\Omega_i\Omega_m) + b_3(\Omega_i\Omega_j\delta_{mn} - \Omega_m\Omega_n\delta_{ij}).\end{aligned} \tag{11}$$

We observe that the axisymmmetry has reduced the number of independent components from 21 to 5 for the pair-symmetric part and from 15 to 3 for the pair-antisymmetric part. This reduction can be explained by considering the reduction of linear tensor representations under rotations (see the Appendix and Ref. [10]). The reduction under rotations is performed by extracting traces, which are rotation invariant *but not* linear invariant. There exists a canonical procedure for doing this trace extraction [14] but we can clarify the procedure by noting that the properties of the expressions in Eq. (10) or Eq. (11) under rotations are determined only by the vector Ω_i (δ_{ij} and ϵ_{ijk} are rotation invariant). For example, the terms with coefficients a_1 and a_4 clearly correspond to scalars ($J=0$), the only terms allowed by isotropy. Furthermore, an expression with the tensor product of n $\mathbf{\Omega}$'s corresponds to the representation $J=n$, usually, with an admixture of lower J representations. Therefore, each coefficient corresponds to a definite J representation, but, in order to obtain the correct tensorial expression of each representation, we need to remove the lower J representations by extracting traces. This induces a linear redefinition of the coefficients within each linear representation.

Finally, we remark that the terms in Eq. (11) with coefficients b_1 and b_2 would not be allowed if isotropy were broken by a polar vector, because the respective terms would be odd under parity [in general, the parity of the representation J associated with a polar vector is $(-)^J$].

A. Traceless components and incompressibility constraint

The preceding tensorial expressions for the viscosity have a part that couples to the velocity divergence u_{ii}. Moreover, they give rise to an isotropic part of the viscous stress tensor σ_{ij} (that is, proportional to δ_{ij}). Therefore, the viscosity tensors must be further decomposed into traceless and trace parts. For incompressible flow we only need the traceless components such that $\eta_{ijkk}=\eta_{kkmn}=0$. They can be extracted by subtracting traces from either η^S_{ijmn} or η^A_{ijmn}. In the general case, that is, with no axial (or any other) symmetry, those tracelessness conditions remove $6+6-1=11$ components (the condition $\eta_{kkll}=0$ appears twice), leaving 25 components. To be more precise, the conditions $\eta^S_{ijkk}=0$ remove six components of η^S_{ijmn} (with $J=2,0$, corresponding to ${}_6S$) and the conditions $\eta^A_{ijkk}=0$ remove the five components of η^A_{ijmn} corresponding to $J=2$.

Indeed, a straightforward calculation yields

$$\begin{aligned}\eta^S_{ijmn}-\tfrac{1}{3}\eta^S_{ijkk}\delta_{mn}-\tfrac{1}{3}\eta^S_{kkmn}\delta_{ij}+\tfrac{1}{9}\eta^S_{kkll}\delta_{ij}\delta_{mn}\\ =a_1(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\tfrac{2}{3}\delta_{ij}\delta_{mn})+a_2[\Omega_i\Omega_m\delta_{jn}\\ +\Omega_j\Omega_m\delta_{in}+\Omega_i\Omega_n\delta_{jm}+\Omega_j\Omega_n\delta_{im}-\tfrac{4}{3}(\Omega_i\Omega_j\delta_{mn}\\ +\Omega_m\Omega_n\delta_{ij})+\tfrac{4}{9}\Omega^2\delta_{ij}\delta_{mn}]+a_3[\Omega_i\Omega_j\Omega_m\Omega_n\\ -\tfrac{1}{3}\Omega^2(\Omega_i\Omega_j\delta_{mn}+\Omega_m\Omega_n\delta_{ij})+\tfrac{1}{9}\Omega^4\delta_{ij}\delta_{mn}],\end{aligned}\tag{12}$$

$$\begin{aligned}\eta^A_{ijmn}-\tfrac{1}{3}\eta^A_{ijkk}\delta_{mn}-\tfrac{1}{3}\eta^A_{kkmn}\delta_{ij}\\ =b_1\Omega_q(\epsilon_{qim}\delta_{jn}+\epsilon_{qin}\delta_{jm}+\epsilon_{qjm}\delta_{in}+\epsilon_{qjn}\delta_{im})\\ +b_2\Omega_q(\epsilon_{qim}\Omega_j\Omega_n+\epsilon_{qin}\Omega_j\Omega_m+\epsilon_{qjm}\Omega_i\Omega_n\\ +\epsilon_{qjn}\Omega_i\Omega_m).\end{aligned}\tag{13}$$

The number of coefficients has been reduced to three for η^S_{ijmn}, corresponding to the $J=4,2,0$ representations, and to two for η^A_{ijmn}, corresponding to $J=3,1$. It is natural that they together constitute the Clebsh-Gordan decomposition of the tensor product of two $J=2$ representations (with dimension $5\times5=25$) [14].

There is another set of tracelessness conditions, namely, $\eta_{ijmj}=0$, but there is no physical reason to impose them. However, note that the six conditions $\eta^S_{ijmj}=0$ remove the $J=2,0$ representations, just leaving $J=4$, while the three conditions $\eta^A_{ijmj}=0$ remove the $J=1$ representation, just leaving $J=3$. Therefore, this last set of tracelessness conditions would select the highest J representations, corresponding to the coefficients a_3 and b_2.

B. Viscosity tensors with antisymmetric pairs

Two crucial assumptions in the reasoning at the beginnning of Sec. III are that the viscous stress tensor is symmetric and that it does not depend on the vorticity [the vorticity tensor is $\omega_{ij}=\partial_{[i}u_{j]}=(\partial_i u_j-\partial_j u_i)/2$]. They lead to a viscosity tensor with symmetry under exchange of indices within the first and second pairs of indices (symmetry by pairs). Those two assumptions are commonly accepted since they are based on basic physical principles: on the one hand, the stress tensor can always be chosen symmetric because of angular momentum conservation; on the other hand, a uniform rotation (leading to a constant vorticity) cannot induce stresses, so a dependence of the stress tensor on vorticity is forbidden. However, both principles, namely, angular momentum conservation and absence of stresses in uniformly rotating fluid, fail in a rotating frame. Therefore, we are allowed to consider viscosity tensors with antisymmetric pairs of indices. We have three types: (i) tensors χ_{ijmn} with the first pair symmetric and the second antisymmetric, which account for an angular momentum conserving coupling to vorticity, (ii) the symmetric type ξ_{ijmn}, that is, tensors with the first pair antisymmetric and the second symmetric, which account for an angular momentum nonconserving coupling to strain rate, and (iii) tensors ζ_{ijmn} with both pairs antisymmetric, which account for an angular momentum nonconserving coupling to vorticity.

The most general tensor with the first pair symmetric and the second antisymmetric has 18 components (see the Appendix). Its axisymmetric form is

$$\begin{aligned}\chi_{ijmn}=(c_1\delta_{ij}+c_2\Omega_i\Omega_j)\epsilon_{lmn}\Omega_l+c_3(\Omega_i\Omega_m\delta_{jn}+\Omega_j\Omega_m\delta_{in}\\ -\Omega_i\Omega_n\delta_{jm}-\Omega_j\Omega_n\delta_{im})+c_4(\epsilon_{imn}\Omega_j+\epsilon_{jmn}\Omega_i).\end{aligned}\tag{14}$$

The constants c_1,c_3,c_2 correspond to $J=1,2,3$, respectively, forming the linear representation ${}_{15}SA$, whereas c_4 corresponds to $J=1$ and ${}_3SA$. Imposing that the tensor be traceless in its first two indices, that is, $\chi_{iimn}=0$, relates the coefficients c_1 and c_2 (the tensor is automatically traceless in the second pair of indices). Therefore, the traceless tensor contains the $J=1,2,3$ representations, corresponding to the Clebsh-Gordan decomposition of the tensor product of the $J=2$ and $J=1$ representations.

There is an analogous axisymmetric structure for the symmetric type ξ_{ijmn}, involving ${}_{15}AS$ and ${}_3AS$, and with coefficients $c'_1,\ldots,c'_4$.

Finally, the tensor ζ_{ijmn} with both pairs antisymmetric has nine components, which the axisymmetry reduces to

$$\begin{aligned}\zeta_{ijmn}=d_1(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})+d_2(\Omega_i\Omega_m\delta_{jn}-\Omega_j\Omega_m\delta_{in}\\ -\Omega_i\Omega_n\delta_{jm}+\Omega_j\Omega_n\delta_{im})+d_3(\epsilon_{imn}\Omega_j-\epsilon_{jmn}\Omega_i).\end{aligned}\tag{15}$$

The constants d_1,d_2 correspond to $J=0,2$, respectively, forming the representation ${}_6A$ (which is pair symmetric), whereas d_3 corresponds to $J=1$ and ${}_3A$ (which is pair antisymmetric: even though it may not seem obvious, $\epsilon_{imn}\Omega_j-\epsilon_{jmn}\Omega_i=-\epsilon_{mij}\Omega_n+\epsilon_{nij}\Omega_m$). The tensor defined by Eq. (15) is trivially traceless in both pairs of indices and corresponds to the Clebsh-Gordan decomposition of the tensor product of two $J=1$ representations.

C. Effective forces associated with the viscosity tensor

The total viscosity tensor $\tau=\eta+\chi+\xi+\zeta$ is defined by

$$\sigma_{ij}=\tau_{ijmn}\partial_m u_n.\tag{16}$$

The force derived from this stress tensor is

$$f_i=\partial\sigma_{ij}/\partial x_j=\tau_{ijmn}\partial_{jm}u_n.\tag{17}$$

The expression that results by substituting the full axisymmetric expression of τ_{ijmn} is fairly complicated: suppressing gradient terms, we obtain

$$\begin{aligned}\boldsymbol{f}=&(a_1-d_1)\nabla^2\boldsymbol{u}-b_1(\boldsymbol{\Omega}\times\nabla^2\boldsymbol{u})-b_2(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})^2(\boldsymbol{\Omega}\times\boldsymbol{u})\\&-(b_2+c_2')(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})(\boldsymbol{\Omega}\times\boldsymbol{\nabla})(\boldsymbol{\Omega}\cdot\boldsymbol{u})+(b_2+c_2)\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\\&\times(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})+(c_4+c_4'+d_3)(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}+(a_2-c_3+c_3'-d_2)\\&\times\boldsymbol{\Omega}\nabla^2(\boldsymbol{\Omega}\cdot\boldsymbol{u})+(a_2+c_3-c_3'-d_2)(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})^2\boldsymbol{u}\\&+a_3\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})^2(\boldsymbol{\Omega}\cdot\boldsymbol{u}).\end{aligned}\tag{18}$$

Several remarks are in order. Note that the fifth and sixth terms of the force involve the vorticity $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{u}$ and are proportional to an odd power of $\boldsymbol{\Omega}$. The terms preceding them are also proportional to an odd power of $\boldsymbol{\Omega}$, except the first one, which is isotropic. The remaining three anisotropic terms, which neither involve the vorticity nor any vector product, are equivalent to the anisotropic force written in Ref. [9]. If we had considered only the tensor η^S to derive the force, we would have obtained precisely these three terms but the first couple of them would have had the same coefficient a_2. As we use the complete tensor τ we have instead that some coefficients are redundant: inspecting Eq. (18), we see that there are two redundant coefficients among c_4, c_4', d_3, two redundant coefficients among a_2, c_3, c_3', d_2, and one redundant coefficient among a_1, d_1.

After taking into account that $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$ and suppressing gradient terms, only remain the coefficients of the part of τ that is traceless in the first and second pair of indices. Gradient terms are longitudinal and the physical force must be transverse (solenoidal); but, after removing these terms, the force is still nontransverse and must be projected with the nonlocal operator $\mathcal{P}$ of Eq. (5). This operation brings back two suppressed gradient terms, namely, $\boldsymbol{\nabla}(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})=0$ and $\boldsymbol{\nabla}[(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})(\boldsymbol{\Omega}\cdot\boldsymbol{u})]$, in addition to producing nonlocal gradient terms.

Finally, we remark that all the terms in Eq. (18) coming from odd-J components of τ, that is, the ones with odd powers of $\boldsymbol{\Omega}$ (with coefficients b_1, b_2, c_2, c_2', c_4, c_4', and d_3), would not be allowed if isotropy were broken by a polar vector.

D. Dissipation

The dissipated power is

$$\begin{aligned}-\int d^3x\boldsymbol{u}\cdot\boldsymbol{f}=-\int d^3x u_i\partial_j\sigma_{ij}=\int d^3x\partial_j u_i\sigma_{ij}\\=\int d^3x u_{ij}\eta_{ijmn}u_{mn}+\int d^3x u_{ij}\chi_{ijmn}\omega_{mn}\\-\int d^3x\omega_{ij}\xi_{ijmn}u_{mn}-\int d^3x\omega_{ij}\zeta_{ijmn}\omega_{mn},\end{aligned}\tag{19}$$

where we have assumed that the velocity vanishes on the boundary to remove the surface integrals, that is, we have assumed that there is no work made by external sources.

As remarked in Ref. [5], η^A does not lead to dissipation; neither does ζ^A. Moreover, if $\chi_{ijmn}=\xi_{mnij}$, the respective terms cancel in Eq. (19). All these nondissipative components of τ do not properly belong to the *viscosity* tensor, although they give rise to forces with dynamical effect. On the other hand, since the dissipation cannot be negative, we can deduce some positivity conditions on the proper coefficients of the viscosity tensor: $a_1>0$, $-d_1>0$, etc.

IV. AXISYMMETRIC FORM OF THE RESPONSE FUNCTION AND VELOCITY CORRELATIONS

It is useful to study the symmetry constraints on the response function and velocity correlations. Here we determine the most general axisymmetric forms of these quantities in the small-wave-number limit (corresponding to large-scale features). The theory of axisymmetric tensors has been the subject of previous analyses of anisotropic turbulence; in particular, it has been treated in papers by Chandrasekhar [6] and by Arad *et al.* [10]. The former uses the old formalism of invariant theory whereas the latter uses the theory of group representations. Unfortunately, both consider only the application to correlation functions in real space, while we are interested here in correlation functions in Fourier space (spectral functions). Therefore, the theory of axisymmetric tensors as is developed in those references must be adapted to Fourier space. Actually, the spectral two-point velocity correlation function in rotating turbulence has already been studied by Cambon and Jacquin [2] and we shall use their results.

A. Axisymmetric form of rank-two tensors

We consider a second-rank tensor that depends on the wave vector $\boldsymbol{k}$ (since we use Fourier space), in addition to the angular velocity $\boldsymbol{\Omega}$. The general form of such a tensor as a linear combination of the tensorial products k_ik_j, $\Omega_i\Omega_j$, $k_i\Omega_j$, Ω_ik_j and the unit tensor δ_{ij} is

$$T_{ij}(\boldsymbol{k})=Ak_ik_j+B\Omega_i\Omega_j+Ck_i\Omega_j+C'\Omega_ik_j+E\delta_{ij},\tag{20}$$

where A,B,C,C',E are arbitrary functions of k, Ω, and $\boldsymbol{k}\cdot\boldsymbol{\Omega}$. However, a more general expression results upon introducing the unit antisymmetric tensor or, equivalently, the vector $\boldsymbol{n}=\boldsymbol{k}\times\boldsymbol{\Omega}$ (assuming that $\boldsymbol{k}$ and $\boldsymbol{\Omega}$ are not parallel) and the corresponding tensor products:

$$\begin{aligned}T_{ij}(\boldsymbol{k})=&Ak_ik_j+Ck_i\Omega_j+Dk_in_j+C'\Omega_ik_j+B\Omega_i\Omega_j+F\Omega_in_j\\&+D'n_ik_j+F'n_i\Omega_j+Gn_in_j.\end{aligned}\tag{21}$$

This expression with nine coefficients is the most general one, because any vector (to be included in a tensor product) can be expressed as a linear combination of $\boldsymbol{k}$, $\boldsymbol{\Omega}$, and $\boldsymbol{n}$.

We remark that expression (20) corresponds to the ordinary quadratic form of Ref. [6], where the terms with the unit antisymmetric tensor are named "skew" forms. This name refers to its reflection (or parity) character: if the two vectors employed in the tensor products of the ordinary quadratic form are polar, this form is parity invariant (even parity), whereas the skew forms change sign under reflections (odd parity) since the vector product is axial. In our case, we begin with a polar vector $\boldsymbol{k}$ and an *axial* vector $\boldsymbol{\Omega}$, so their vector product $\boldsymbol{n}$ is polar. Hence, the terms of Eq. (21) that change sign under reflections are the ones with only one $\boldsymbol{\Omega}$.

Instead of the basis formed by $\boldsymbol{k}$, $\boldsymbol{\Omega}$, and $\boldsymbol{n}$, it may be more convenient to use an orthonormal basis. Any couple of linearly independent vectors determine an orthonormal basis, in particular, the two vectors $\boldsymbol{k}$ and $\boldsymbol{\Omega}$ lead to the one given by $\boldsymbol{k}/k$, $\boldsymbol{e}^{(1)}=\boldsymbol{k}\times\boldsymbol{\Omega}/|\boldsymbol{k}\times\boldsymbol{\Omega}|$, and $\boldsymbol{e}^{(2)}=\boldsymbol{k}\times\boldsymbol{e}^{(1)}/|\boldsymbol{k}\times\boldsymbol{e}^{(1)}|$ [2]. Note that the vector $\boldsymbol{\Omega}$ is axial, so $\boldsymbol{e}^{(1)}$ is polar but $\boldsymbol{e}^{(2)}$ is axial. Interchanging the role of $\boldsymbol{\Omega}$ and $\boldsymbol{k}$, we get a different orthonormal basis, with the vector $\boldsymbol{e}^{(1)}$ in common: $\boldsymbol{e}^3=\boldsymbol{\Omega}/\Omega$, $\boldsymbol{e}^1=\boldsymbol{k}\times\boldsymbol{\Omega}/|\boldsymbol{k}\times\boldsymbol{\Omega}|=\boldsymbol{e}^{(1)}$, and $\boldsymbol{e}^2=\boldsymbol{\Omega}\times\boldsymbol{e}^1/|\boldsymbol{\Omega}\times\boldsymbol{e}^1|$. This basis (which we denote by superindices without parentheses) is more adequate. Note that both $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ are polar.

Any rank-two tensor can be expressed in the latter basis as

$$T_{ij}=\tilde{T}_{pq}e_i^p e_j^q. \tag{22}$$

There are three pieces in T_{ij} that are independent under rotations: the trace, the antisymmetric part, and the traceless symmetric part. This is the Clebsch-Gordan decomposition of the vector tensor product into the irreducible representations $J=0$, $J=1$, and $J=2$ of the rotation group. However, to classify the behavior of the components of the second-rank tensor under rotations around $\boldsymbol{\Omega}$, that is, under the two-dimensional rotation subgroup O(2) of the full rotation group O(3), it is best to use the given basis (components $\tilde{T}_{pq}$). The irreducible one-dimensional representations of O(2) are complex, labeled by an integer M $(-J\leqslant M\leqslant J)$. The real irreducible representations are labeled by $|M|$ and are two dimensional (except the scalar $M=0$ representation) [14]. We have that $\boldsymbol{e}^3$ is a scalar, and $\{\boldsymbol{e}^1,\boldsymbol{e}^2\}$ form the real $|M|=1$ representation. Consequently, $\tilde{T}_{33}$ is the scalar $M=0$ representation, $\tilde{T}_{13},\tilde{T}_{23},\tilde{T}_{31},\tilde{T}_{32}$ belong to two $|M|=1$ representations, and the remaining components in the 2×2 block matrix can be subdivided into its trace $(M=0)$, its antisymmetric part $(M=0)$, and its traceless symmetric part $(|M|=2)$. Furthermore, it is not difficult to ascribe each M representation to a definite J representation.

B. Axisymmetric form of the response function

The response function is defined by

$$G_{ij}(\boldsymbol{k},\omega)=\left.\frac{\delta\langle u_i(\boldsymbol{k},\omega)\rangle}{\delta f_j(\boldsymbol{k},\omega)}\right|_{\boldsymbol{f}=\boldsymbol{0}} \tag{23}$$

(introducing a nonrandom part in the external forcing $\boldsymbol{f}$). So we can write, at linear order in $\boldsymbol{f}$,

$$\langle u_i(\boldsymbol{k},\omega)\rangle=G_{ij}(\boldsymbol{k},\omega)f_j(\boldsymbol{k},\omega). \tag{24}$$

Conversely,

$$G_{ij}^{-1}(\boldsymbol{k},\omega)\langle u_j(\boldsymbol{k},\omega)\rangle=f_i(\boldsymbol{k},\omega), \tag{25}$$

which tells us, on account of Eq. (17), that the quadratic term in the expansion of $G_{ij}^{-1}(\boldsymbol{k},0)$ in powers of $\boldsymbol{k}$ is related with the viscosity tensor. To be precise, we have that

$$g_{ijmn}:=-\frac{1}{2}\left.\frac{\partial^2 G_{ij}^{-1}}{\partial k_m\partial k_n}\right|_{\boldsymbol{k}=\boldsymbol{0}}=\frac{1}{2}(\tau_{imjn}+\tau_{injm}), \tag{26}$$

defining what we call the (four-rank) response tensor. This tensor has symmetry in the pair mn, so it has 54 independent components, while τ (in the generic case) has 81 components. Indeed, the 27 components of the tensor $\frac{1}{2}(\tau_{imjn}-\tau_{injm})$ do not contribute to the response function.

1. Mapping the viscosity tensor to the response tensor

We can take in Eq. (26) η,χ,ξ or ζ for τ. On the other hand, g_{ijmn} can be decomposed into ij-symmetric and ij-antisymmetric parts, corresponding to the respective parts of the response matrix. Therefore, g_{ijmn} has $6\times6=36$ components with symmetry in both pairs (belonging to the S representation) and $3\times6=18$ components with antisymmetry in the first pair and symmetry in the second pair (belonging to the AS representation).

Let us first analyze the components of g coming from η^S. We note that $g_{ijmn}=(\eta^S_{imjn}+\eta^S_{injm})/2=(\eta^S_{jnim}+\eta^S_{jmin})/2=g_{jimn}$. Furthermore, this tensor is pair symmetric: $g_{mnij}=(\eta^S_{minj}+\eta^S_{mjni})/2=(\eta^S_{imjn}+\eta^S_{jmin})/2=(\eta^S_{imjn}+\eta^S_{injm})/2=g_{ijmn}$. So the 21 pair-symmetric components of η^S (representations ${}_{15}S$ and ${}_6S$) are transformed by Eq. (26) into the 21 pair-symmetric components of g; in particular, the totally symmetric representation ${}_{15}S$ of η^S is left invariant. Given that we can substitute η^S by ζ^S in the preceding equations, we conclude that the six pair-symmetric components of ζ^S $({}_6A)$ are transformed by Eq. (26) into six pair-symmetric components of g (linear combinations of ${}_{15}S$ and ${}_6S$).

We also note that if we take χ for τ in Eq. (26) and symmetrize in ij, the tensor $g_{ijmn}=\frac{1}{4}(\chi_{imjn}+\chi_{injm}+\chi_{jmin}+\chi_{jnim})$ is pair antisymmetric, owing to the symmetry of χ. An analogous property is fulfilled by the tensor g_{ijmn} constructed in the same way from ξ. So the 15 components of χ from ${}_{15}SA$ or the 15 components of ξ from ${}_{15}AS$ are transformed by Eq. (26) into the 15 pair-antisymmetric components of g (representation ${}_{15}S'$). On the other hand, it can be proved that χ or ξ belonging to representations ${}_3SA$ or ${}_3AS$, respectively, yield vanishing g (they contribute instead to $\tau_{imjn}-\tau_{injm}$).

As regards the ij-antisymmetric part of g_{ijmn}, note that $g_{ijmn}=(\eta^A_{imjn}+\eta^A_{injm})/2=(-\eta^A_{jnim}-\eta^A_{jmin})/2=-g_{jimn}$. So the 15 components of η^A (representation ${}_{15}S'$) are transformed by Eq. (26) into 15 components of g_{ijmn} with antisymmetry in ij $({}_{15}AS)$. Given that we can substitute η^A by ζ^A in the preceding equations, the remaining three components of ζ^A $({}_3A)$ are transformed into three components of g_{ijmn} forming the representation ${}_3AS$. Finally, we also have the mapping $SA\rightarrow AS$ given by $g_{ijmn}=\frac{1}{4}(\chi_{imjn}+\chi_{injm}-\chi_{jmin}-\chi_{jnim})$ and a similar mapping $AS\rightarrow AS$.

2. Axisymmetric form of the response tensor

The preceding mapping has been established with full generality, without considering any particular spatial symmetry. If we take axisymmetry into account, Eq. (26) provides the linear relations between the coefficients in the axisymmetric form of g_{ijmn} and the coefficients in the axisymmetric forms of η, χ, ξ or ζ. The 54 components of g_{ijmn} belong to the S and AS representations, therefore, g_{ijmn} has $8+4=12$ coefficients: the axisymmetric form of the components g_{ijmn} with symmetry in the first pair is like the forms of η in Eqs. (10) and (11), with other coefficients, say $\alpha_1,\ldots,\alpha_5$, β_1,β_2,β_3; the axisymmetric form of the components g_{ijmn} with antisymmetry in the first pair is like the form of ξ, with other coefficients, say $\gamma_1,\ldots,\gamma_4$.

The above-mentioned coefficients in the response tensor can also be obtained by expanding in powers of $\boldsymbol{k}$ the axisymmetric expression of G_{ij}^{-1} (21). Considering that the unit antisymmetric tensor does not appear in the tensors with coefficients $\alpha_1,\ldots,\alpha_5$, β_3, γ_3, the corresponding part of G_{ij}^{-1} is given just by expression (20). Then the coefficients $\alpha_1,\ldots,\alpha_5$, β_3, γ_3 must arise by expanding A,B,C,C',E in powers of $\boldsymbol{k}$ such that the total expression is of second degree in $\boldsymbol{k}$. In particular, $B=B_1k^2+B_2(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2$ and $E=E_1k^2+E_2(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2$, while A is a constant and the coefficient functions of the terms that are of first degree in $\boldsymbol{k}$, namely, C,C', can only be expanded up to the first order (proportional to $\boldsymbol{k}\cdot\boldsymbol{\Omega}$). Therefore, this expansion just doubles the coefficients of $\Omega_i\Omega_j$ and δ_{ij}, producing seven coefficients altogether. We can divide them between the six coefficients arising from the symmetric $G_{(ij)}^{-1}$ $(C=C')$ and the one corresponding to the antisymmetric $G_{[ij]}^{-1}$, namely, $C-C'$.

The part of G_{ij}^{-1} that includes the unit antisymmetric tensor, with coefficients D,D',F,F', and G in Eq. (21), corresponds to $\beta_1,\beta_2,\gamma_1,\gamma_2$, and γ_4. They can be divided into β_1,β_2 for the symmetric $G_{(ij)}^{-1}$ and γ_1,γ_2, and γ_4 for the antisymmetric $G_{[ij]}^{-1}$.

3. Higher-rank axisymmetric response tensors

An expansion of G_{ij}^{-1} in powers of $\boldsymbol{k}$ to orders higher than the quadratic order yields response tensors similar to g_{ijmn} in Eq. (26) but of higher rank. Given that G_{ij}^{-1} must be parity symmetric, only even powers of $\boldsymbol{k}$ can appear. For example, the next higher-rank response tensor g_{ijmnpq} is symmetric in the last four indices and, therefore, has $9\times 15=135$ components, but this number is reduced by the axisymmetry. The following higher-rank response tensors are progressively more complex, of course.

C. Axisymmetric form of the two-point velocity correlation

Let us introduce the spectral two-point velocity correlation

$$\langle u_i(\boldsymbol{k},\omega)\,u_j(\boldsymbol{k}',\omega')\rangle=(2\pi)^4\mathcal{U}_{ij}(\omega,\boldsymbol{k})\,\delta(\omega+\omega')\,\delta^3(\boldsymbol{k}+\boldsymbol{k}'). \tag{27}$$

We have that $\mathcal{U}_{ij}(\omega,\boldsymbol{k})=\mathcal{U}_{ji}(-\omega,-\boldsymbol{k})$. Furthermore, transversality implies that $k_i\mathcal{U}_{ij}=k_i\mathcal{U}_{ji}=0$. So, in the isotropic case, the spectral two-point velocity correlation is given in terms of only one function:

$$\mathcal{U}_{ij}(\omega,\boldsymbol{k})=\mathcal{P}_{ij}(\boldsymbol{k})\mathcal{U}(\omega,\boldsymbol{k}). \tag{28}$$

Taking into account that equal-time correlations are more useful, let us define

$$\mathcal{U}_{ij}(\boldsymbol{k})=\int\frac{d\omega}{2\pi}\mathcal{U}_{ij}(\omega,\boldsymbol{k}) \tag{29}$$

(assuming that the integral is convergent), so that

$$\langle u_i(\boldsymbol{k},t)u_j(\boldsymbol{k}',t)\rangle=(2\pi)^3\mathcal{U}_{ij}(\boldsymbol{k})\,\delta^3(\boldsymbol{k}+\boldsymbol{k}'). \tag{30}$$

As demonstrated in Sec. IV A, the general axisymmetric rank-two tensor has nine independent coefficient functions, but the transversality conditions reduce their number. The number of independent conditions is five, so just four coefficient functions remain independent, namely, the ones corresponding to the tensor products of the transverse vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$. It is convenient to use the basis corresponding to circular polarizations $\boldsymbol{N}=\boldsymbol{e}^{(1)}-i\boldsymbol{e}^{(2)}$, $\boldsymbol{N}^*=\boldsymbol{e}^{(1)}+i\boldsymbol{e}^{(2)}$, so that the resulting tensor can be written as [2]

$$\mathcal{U}_{ij}(\boldsymbol{k})=e(\boldsymbol{k})\mathcal{P}_{ij}+\mathrm{Re}[z(\boldsymbol{k})N_iN_j]+ih(\boldsymbol{k})\epsilon_{ijl}\frac{k_l}{k^2}. \tag{31}$$

The quantities $e(\boldsymbol{k})$ and $h(\boldsymbol{k})$ are the energy and helicity spectrum, and $z(\boldsymbol{k})$ is a "complex deviator." They all are even functions of $\boldsymbol{k}$. The preceding form is equivalent to the form with the products $e_i^{(1)}e_j^{(2)}$, on account that $N_iN_j=e_i^{(1)}e_j^{(1)}-e_i^{(2)}e_j^{(2)}-i(e_i^{(1)}e_j^{(2)}+e_i^{(2)}e_j^{(1)})$, $\mathcal{P}_{ij}=e_i^{(1)}e_j^{(1)}+e_i^{(2)}e_j^{(2)}$, and $\epsilon_{ijl}k_l/k=e_i^{(1)}e_j^{(2)}-e_i^{(2)}e_j^{(1)}$.

Velocity correlations for more than two points lend themselves to be expressed in similar though more complicated ways.

V. CONNECTION WITH SOME APPROACHES TO ANISOTROPIC TURBULENCE

We have already mentioned that fourth-rank tensors associated with anisotropic turbulence (but without rotation) have been studied before; for example, in Ref. [7]. More recently, in Ref. [11] one fourth-rank tensor for a flow with a constant strain rate has been defined. Another fourth-rank tensor is defined in Ref. [12] in connection with the linearization of a closure equation in the presence of weak anisotropy. We now explore connections between our anisotropic viscosity tensor and those tensors.

The fourth-rank tensor $C_{ijmn}(\boldsymbol{k})$ of Ref. [11] expresses proportionality between the contribution to the correlation $\mathcal{U}_{ij}(\boldsymbol{k})$ [defined in Eq. (29)] from anisotropy and the constant strain rate producing the anisotropy:

$$\delta\mathcal{U}_{ij}(\boldsymbol{k})=C_{ijmn}(\boldsymbol{k})u_{mn}, \tag{32}$$

where the strain rate u_{mn} is constant. The Reynolds stress tensor

$$\langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t)\rangle = \int \frac{d^3k}{(2\pi)^3}\mathcal{U}_{ij}(\boldsymbol{k}) = \mathcal{U}_{ij} \tag{33}$$

has a deviatoric part that, according to Eq. (32), is proportional to the strain rate, the proportionality constant being the integral of the tensor $C_{ijmn}(\boldsymbol{k})$:

$$\delta\mathcal{U}_{ij} = \int \frac{d^3k}{(2\pi)^3}\,\delta\mathcal{U}_{ij}(\boldsymbol{k}) = \int \frac{d^3k}{(2\pi)^3}C_{ijmn}(\boldsymbol{k})u_{mn}. \tag{34}$$

It is also possible to assume that the Reynolds stress tensor (33) and the strain rate have some mild dependence on the spatial coordinate $\boldsymbol{x}$. The corresponding generalization of Eq. (34) is a phenomenological (mean-field) closure relation that can be justified with a multiscale method applied to the Navier-Stokes equation linearized with respect to the $\boldsymbol{x}$-dependent (large-scale) mean flow [7]. Comparing this mean-field relation with Eq. (8), we deduce a relation between our viscosity tensor η_{ijmn} and $C_{ijmn}(\boldsymbol{k})$, namely,

$$\eta_{ijmn} = \rho\int \frac{d^3k}{(2\pi)^3}C_{ijmn}(\boldsymbol{k}). \tag{35}$$

We must note, however, that the form of $C_{ijmn}(\boldsymbol{k})$ in terms of projectors $\mathcal{P}_{ij}(\boldsymbol{k})$ proposed in Ref. [11] leads to the usual isotropic η_{ijmn}. Indeed, one needs an additional quantity, such as the vector $\boldsymbol{\Omega}$, to define an anisotropic viscosity.

More sophisticated closure schemes involve relations between the three and two-point velocity correlation functions. The Navier-Stokes equation leads to an equation involving these two types of correlations, first derived by von Kármán and Howarth assuming isotropy. Chandrasekhar [6] developed a theory of axisymmetric tensors to generalize this equation to axisymmetric turbulence. As remarked by Frisch [8], it is easy to derive a fully anisotropic version of the Kármán-Howarth equation, which he calls the Kármán-Howarth-Monin equation. In Ref. [12], the Fourier transform of this equation is used as the basis of a closure scheme related with the direct interaction approximation, in which the function $\mathcal{U}_{ij}(\boldsymbol{k})$ satisfies (in stationary conditions) a nonlinear equation:

$$D_{mn}(\boldsymbol{k}) \equiv I_{mn}(\boldsymbol{k}) - \nu k^2\mathcal{U}_{mn}(\boldsymbol{k}), \tag{36}$$

where I_{mn} is an integral operator quadratic in $\mathcal{U}_{ij}$. In addition, we have introduced an external random forcing, absent in Ref. [12], which is Gaussian *and* white in time and, hence, is represented by the spectral two-point correlation $D_{mn}(\boldsymbol{k})$ (see Ref. [8] for a general description of closure equations). If the molecular viscosity ν vanishes, the external forcing is not necessary, as in Ref. [12]. However, the introduction of D_{mn} allows us to substitute the four-rank tensor defined in Ref. [12] by a four-rank tensor more useful to connect with the viscosity tensor.

If the forcing is isotropic, we expect that Eq. (36) has isotropic solutions. One can then linearize this equation around an isotropic solution, namely,

$$\delta D_{mn}(\boldsymbol{q}) = \left.\frac{\delta D_{mn}(\boldsymbol{q})}{\delta\mathcal{U}_{ij}(\boldsymbol{k})}\right|_{\mathcal{U}_{ij}=e\mathcal{P}_{ij}}\delta\mathcal{U}_{ij}(\boldsymbol{k}), \tag{37}$$

where δD_{mn} represents an anisotropic perturbation of an isotropic forcing such that $D_{mn} = D\mathcal{P}_{mn}$ [note that isotropy implies that $\mathcal{U}_{ij} = e\mathcal{P}_{ij}$, according to Eq. (31)]. The solution of this linear equation is obtained by inverting the matrix of pairs of indices, deriving a sort of tensorial response function,

$$G_{ijmn}(\boldsymbol{k},\boldsymbol{q}) = \left.\frac{\delta\mathcal{U}_{ij}(\boldsymbol{k})}{\delta D_{mn}(\boldsymbol{q})}\right|_{D_{mn}=D\mathcal{P}_{mn}}, \tag{38}$$

which measures the response to the anisotropic perturbation. Considering the role of the molecular kinematic viscosity ν in Eq. (36), we can tentatively define a kinematic viscosity tensor as

$$\nu_{ijmn} = -\frac{1}{2}\frac{\partial^2}{\partial k_l\partial k_l}\int d^3q\,G^{-1}_{ijmn}(\boldsymbol{k},\boldsymbol{q})\big|_{\boldsymbol{k}=\boldsymbol{0}}. \tag{39}$$

This relation between a tensorial viscosity and a response tensor is an alternative to Eq. (26), valid when we replace the original Navier-Stokes equation with the closure Eq. (36). However, the actual computation of ν_{ijmn} necessarily leads to an isotropic tensor, since there is nothing in Eq. (38) capable of breaking rotation invariance.

VI. CONCLUSIONS

We have applied symmetry principles to homogeneous turbulence subjected to uniform rotation, focusing on the four-rank tensor defining the linear relation between the stress tensor and the velocity derivatives, which we call the viscosity tensor. The most general tensor comprises five parts.

(a) A tensor η^S symmetric by pairs of indices and pair symmetric, accounting for the usual proportionality relation between the (anisotropic) stress and the strain rate.

(b) A tensor η^A symmetric by pairs and pair antisymmetric embodying a new relation between the stress and the strain rate, typical of rotating fluids, since it does not lead to dissipation.

(c) A tensor χ symmetric in the first pair of indices and antisymmetric in the second, which accounts for a stress tensor coupling to vorticity.

(d) A tensor ξ antisymmetric in the first pair of indices and symmetric in the second, which accounts for the antisymmetric part of the stress tensor (angular momentum nonconserving) that couples to the strain rate.

(e) A tensor ζ antisymmetric in both pairs of indices which accounts for the antisymmetric part of the stress tensor that couples to the vorticity. This tensor can be further decomposed into pair-symmetric and pair-antisymmetric parts, like η.

Group theory helps us to find the linearly independent components (as representations of the linear group) in each part. Every part adopts a particular axisymmetric form, which can be deduced from the examination of the decomposition of the linearly independent components under rotations.

This variety of components of the viscosity tensor is reflected in the various effective forces that arise from them. Some of these are longitudinal, that is, they are the gradient of a potential, and therefore do not contribute to the transverse rotating fluid equation. However, they arise from the turbulent state and contribute to the equation that determines the equilibrium state, so they may have practical relevance. Already at first order in Ω, we have the potential $\mathbf{\Omega}\cdot\boldsymbol{\omega}$, which reminds us of the spin-orbit coupling of atomic physics. At higher orders in Ω, we find more complicated potentials.

Although the most general four-rank viscosity tensor includes terms that lead to the stress coupling to vorticity and not conserving angular momentum, one may wonder if they are really necessary. If we take the criterion of having the most general axisymmetric *transverse* force, we could remove redundant coefficients in Eq. (18): this equation has nine terms but includes the 14 coefficients of the traceless viscosity tensor. For example, we could remove all the coefficients belonging to ζ, and the couple c_3, c_4 (belonging to χ) or the couple c_3', c_4' (belonging to ξ), but not c_2 or c_2'. Hence, we conclude that the only part of the viscosity tensor that can be neglected is ζ, but χ and ξ must be present. So we still have that the stress couples to vorticity and does not conserve angular momentum (in the rotating frame).

Finally, in our analysis of the four-rank tensors defined in some approaches to nonrotating anisotropic turbulence, we have seen there are similarities with our viscosity tensor in their definition and, therefore, that the respective definitions can be connected. However, the lack of specification of a quantity that breaks rotation invariance precludes that the actual values of the tensors corresponding to nonrotating turbulence can be anisotropic. It is possible, nevertheless, to provide such a symmetry-breaking quantity: for example, an anisotropic noise spectral correlation D_{mn}. In particular, one can introduce an axisymmetric D_{mn} by postulating the presence of a global vector (of unknown origin), as in the perturbative approach of Ref. [9]. If this symmetry-breaking vector were axial instead of polar, the four-rank tensor ν_{ijmn} of Sec. V should have the same form that our η_{ijmn}.

ACKNOWLEDGMENTS

I thank C. Molina-París for conversations. My work was supported by a Ramón y Cajal contract and by Grant No. BFM2002-01014, both of the Ministerio de Ciencia y Tecnología.

APPENDIX: RESOLUTION OF THE GENERAL FOUR-RANK TENSOR BY THE SYMMETRY OF PAIRS OF INDICES

Let us work out first the resolution of the general four-rank tensor T_{ijmn} into a sum of tensors of definite symmetry type given by standard Young tableaux. Young tableaux indicate certain symmetry operations performed on the indices [14]. We can consider the general four-rank tensor as a tensorial product of four vectors and, therefore, write its resolution as the Clebsch-Gordan decomposition for the linear group GL(3) of the corresponding direct product:

$$\begin{ytableau} i \end{ytableau} \otimes \begin{ytableau} j \end{ytableau} \otimes \begin{ytableau} m \end{ytableau} \otimes \begin{ytableau} n \end{ytableau} = \begin{ytableau} i & j & m & n \end{ytableau} \oplus \begin{ytableau} i & j & m \\ n \end{ytableau} \oplus \begin{ytableau} i & j & n \\ m \end{ytableau} \oplus \begin{ytableau} i & m & n \\ j \end{ytableau} \oplus \begin{ytableau} i & j \\ m & n \end{ytableau} \oplus \begin{ytableau} i & m \\ j & n \end{ytableau} \oplus \begin{ytableau} i & j \\ m \\ n \end{ytableau} \oplus \begin{ytableau} i & m \\ j \\ n \end{ytableau} \oplus \begin{ytableau} i & n \\ j \\ m \end{ytableau} \tag{A1}$$

The dimensions of the GL(3) representations on the right-hand side are the following: 15 for the totally symmetric representation, 15 for the next mixed symmetry representation, 6 for the following mixed symmetry representation, and 3 for the last mixed symmetry representation. Therefore, we have $81 = 15 + 3\times 15 + 2\times 6 + 3\times 3$.

We intend to show the correspondence of the preceding tensorial representations with the tensorial representations selected according to the symmetry relative to pairs of indices. These are constructed as direct product of representations:

$$\begin{ytableau} i & j \end{ytableau} \otimes \begin{ytableau} m & n \end{ytableau}, \quad \begin{ytableau} i \\ j \end{ytableau} \otimes \begin{ytableau} m & n \end{ytableau}, \quad \begin{ytableau} i & j \end{ytableau} \otimes \begin{ytableau} m \\ n \end{ytableau}, \quad \begin{ytableau} i \\ j \end{ytableau} \otimes \begin{ytableau} m \\ n \end{ytableau}. \tag{A2}$$

The corresponding dimensions are the following: 36 for the symmetric-symmetric, 18 for both the antisymmetric-symmetric and the symmetric-antisymmetric, and 9 for the antisymmetric-antisymmetric. The symmetric-symmetric tensor and the antisymmetric-antisymmetric tensor can be further resolved into pair-symmetric and pair-antisymmetric components. Moreover, most of the six resulting representations are still reducible. To find the irreducible representations, we will take advantage of the Clebsch-Gordan decomposition given by Eq. (A1), superimposing on it the symmetry relative to pairs of indices.

To have symmetry in the first pair, say, we may just symmetrize the general tensor (A1) over indices ij. This operation immediately removes the fourth, sixth, eighth, and ninth representations, which are antisymmetric in ij. In other words, given that

$$\boxed{i}\otimes\boxed{j} = \begin{array}{|c|c|}\hline i&j\\\hline\end{array} \oplus \begin{array}{|c|}\hline i\\\hline j\\\hline\end{array}\,, \tag{A3}$$

we can resolve the general tensor (A1) into ij-symmetric and ij-antisymmetric parts. The former is

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array}\otimes\boxed{m}\otimes\boxed{n} = \begin{array}{|c|c|c|c|}\hline i&j&m&n\\\hline\end{array}\oplus\left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline i&j\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}\right]_{(ij)}\,, \tag{A4}$$

where the right-hand side of the latter equation is the result of symmetrization over indices ij (unnecessary in the totally symmetric representation). The ij-antisymmetric part is more complicated but we will not need it.

We can further resolve Eq. (A4) by symmetrizing or antisymmetrizing over the remaining pair of indices:

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline m&n\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline i&j&m&n\\\hline\end{array}\oplus\left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\end{array}\right]_{(ij),(mn)}\oplus\left[\begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array}\right]_{(ij),(mn)}\,, \tag{A5}$$

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array}\otimes\begin{array}{|c|}\hline m\\\hline n\\\hline\end{array} = \left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\end{array}\right]_{(ij),[mn]}\oplus\left[\begin{array}{|c|c|}\hline i&j\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}\right]_{(ij),[mn]}\,. \tag{A6}$$

Let us analyze Eq. (A5). Using the definitions of the Young tableaux, we compute

$$\left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}\\\cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}\\\cline{1-1}\end{array}\right]_{(ij),(mn)} = T_{ijmn}+T_{jimn}+T_{ijnm}+T_{jinm}-T_{mnij}-T_{mnji}-T_{nmij}-T_{nmji}\,, \tag{A7}$$

$$\begin{aligned}\left[\begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array}\right]_{(ij),(mn)} = {} & T_{ijmn}+T_{jimn}+T_{ijnm}+T_{jinm}+T_{mnij}+T_{mnji}+T_{nmij}+T_{nmji}\\ & +\frac{1}{2}\left[-T_{mjin}-T_{mijn}-T_{jmin}-T_{imjn}-T_{mjni}-T_{minj}-T_{jmni}-T_{imnj}\right.\\ & \left.-T_{inmj}-T_{jnmi}-T_{nimj}-T_{njmi}-T_{injm}-T_{jnim}-T_{nijm}-T_{njim}\right]. \end{aligned} \tag{A8}$$

If we denote the components of the tensor with symmetry by pairs [given by the left-hand side of Eq. (A5)] as

$$S_{ijmn}=T_{ijmn}+T_{jimn}+T_{ijnm}+T_{jinm}\,, \tag{A9}$$

we can write Eq. (A7) as

$${}_{15}S'_{ijmn}=S_{ijmn}-S_{mnij}$$

and Eq. (A8) as

$${}_{6}S_{ijmn}=S_{ijmn}+S_{mnij}-\tfrac{1}{2}[S_{imjn}+S_{inmj}+S_{mjin}+S_{njmi}],$$

where the left subscript indicates the dimension of the irreducible linear representations. On the other hand, the totally symmetric representation reads

$${}_{15}S_{ijmn}=S_{ijmn}+S_{mnij}+S_{jmin}+S_{imjn}+S_{injm}+S_{jnim}\,. \tag{A10}$$

Now, let us analyze Eq. (A6). We compute

$$
\left[\begin{array}{|c|c|c|}\hline i & j & m \\ \hline n \\ \cline{1-1} \end{array} + \begin{array}{|c|c|c|}\hline i & j & n \\ \hline m \\ \cline{1-1} \end{array} \right]_{(ij),[mn]} = T_{ijmn} + T_{jimn} - T_{ijnm} - T_{jinm}
$$
$$
+ \frac{1}{2}[-T_{injm} - T_{imnj} - T_{jnim} - T_{jmni} - T_{mjni} - T_{minj} - T_{njim} - T_{nijm}
$$
$$
+ T_{inmj} + T_{imjn} + T_{jnmi} + T_{jmin} + T_{mijn} + T_{mjin} + T_{nimj} + T_{njmi}], \tag{A11}
$$

$$
\left[\begin{array}{|c|c|}\hline i & j \\ \hline m \\ \cline{1-1} n \\ \cline{1-1} \end{array} \right]_{(ij),[mn]} = T_{ijmn} + T_{jimn} - T_{ijnm} - T_{jinm}
$$
$$
+ \frac{1}{2}[T_{mjni} + T_{minj} + T_{njim} + T_{nijm} - T_{mjin} - T_{mijn} - T_{njmi} - T_{nimj}
$$
$$
+ T_{jmni} + T_{imnj} + T_{jnim} + T_{injm} - T_{jmin} - T_{imjn} - T_{jnmi} - T_{inmj}]. \tag{A12}
$$

If we denote the components of the tensor with symmetry in the first pair and antisymmetry in the second [given by the left-hand side of Eq. (A6)] as

$$
SA_{ijmn} = T_{ijmn} + T_{jimn} - T_{ijnm} - T_{jinm}, \tag{A13}
$$

we can write Eq. (A11) as

$$
{}_{15}SA_{ijmn} = SA_{ijmn} + \tfrac{1}{2}[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}].
$$

and Eq. (A12) as

$$
{}_{3}SA_{ijmn} = SA_{ijmn} - \tfrac{1}{2}[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}].
$$

In analogy with Eq. (A13), we define

$$
AS_{ijmn} = T_{ijmn} - T_{jimn} + T_{ijnm} - T_{jinm}, \tag{A14}
$$

and we have analogous definitions for the irreducible linear representations ${}_{15}AS_{ijmn}$ and ${}_{3}AS_{ijmn}$.

We must also define the tensor with antisymmetry by pairs

$$
A_{ijmn} = T_{ijmn} - T_{jimn} - T_{ijnm} + T_{jinm}, \tag{A15}
$$

which can be divided into pair-symmetric and pair-antisymmetric tensors as

$$
{}_{6}A_{ijmn} = A_{ijmn} + A_{mnij}, \tag{A16}
$$

$$
{}_{3}A_{ijmn} = A_{ijmn} - A_{mnij}. \tag{A17}
$$

Note that ${}_{6}A$ exactly corresponds to the second square Young tableau of Eq. (A1).

Of course, many of the preceding irreducible representations of the group GL(3) are reducible with respect to its rotation subgroup O(3), since the metric δ_{ij} allows one to extract lower-rank representations by contracting indices. For example, the totally symmetric tensor contains the representations $J=4,2,0$ [14]. The six-dimensional representation given by ${}_{6}S$ or ${}_{6}A$ contain $J=2,0$. The 15-dimensional representations given by ${}_{15}S'$ or ${}_{15}SA$ contain $J=3,2,1$. The irreducible representations of the group O(3) are the symmetric traceless tensors [14]. Therefore, it must be possible to express each of the previous representations in terms of four-rank tensors as symmetric traceless tensors (of lower rank). For example, for $J=3$, from ${}_{15}S'$ we obtain $T_{ljn} = \epsilon_{lim}S'_{ijmn} + \epsilon_{jim}S'_{ilmn} + \epsilon_{nim}S'_{ijml}$; for $J=3$, from ${}_{15}SA$ we obtain $T_{pij} = \epsilon_{pmn}SA_{ijmn} + \epsilon_{imn}SA_{jpmn} + \epsilon_{jmn}SA_{pimn}$. Note that, in both cases, the components of the $J=2$ representations vanish and do not contribute to the third-rank tensor, while the remaining $J=1$ representation must be removed from this tensor by imposing that it be traceless.

[1] H.P. Greenspan, *The Theory of Rotating Fluids* (Cambridge University Press, Cambridge 1968), reissued by Breukelen Press, 1990.

[2] C. Cambon and L. Jacquin, J. Fluid Mech. **202**, 295 (1989).

[3] L. Jacquin, O. Leuchter, C. Cambon, and J. Mathieu, J. Fluid Mech. **220**, 1 (1990).

[4] C. Cambon, N.N. Mansour, and F.S. Godeferd, J. Fluid Mech. **337**, 303 (1997).

[5] J. Gaite, D. Hochberg, and C. Molina-París, Phys. Rev. E **67**, 026304 (2003).

[6] S. Chandrasekhar, Philos. Trans. R. Soc. London, Ser. A **242**, 442 (1950); Proc. R. Soc. London, Ser. A **203**, 358 (1950).

[7] B. Dubrulle and U. Frisch, Phys. Rev. A **43**, 5355 (1991).

[8] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cam-

bridge University Press, Cambridge 1995).
[9] J. Busa, M. Hnatich, J. Honkonen, and D. Horvath, Phys. Rev. E **55**, 381 (1997).
[10] I. Arad, V.S. L'vov, and I. Procaccia, Phys. Rev. E **59**, 6753 (1999).
[11] T. Ishihara, K. Yoshida, and Y. Kaneda, Phys. Rev. Lett. **88**, 154501 (2002).
[12] V.S. L'vov, I. Procaccia, and V. Tiberkevich, Phys. Rev. E **67**, 026312 (2003).
[13] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).
[14] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover, New York, 1989).
[15] We use the common notation for symmetrization or antisymmetrization over two indices: $A_{(ij)}=(A_{ij}+A_{ji})/2$ and $A_{[ij]}=(A_{ij}-A_{ji})/2$.
[16] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity* (Pergamon Press, Oxford, 1986).